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A THEOREM OF THE TRANSVERSAL THEORY FOR MATROIDS OF FINITE CHARACTER

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Let $M=(S,I)$ be a matroid of finite character on the infinite set S. Let $\mathscr{A}=\langle A_i : i\in I\rangle$ be any system of subsets of S each having finite rank and let $\mathscr{B}=\langle B_j : j\in J\rangle$ be a finite system of sets of arbitrary rank. Necessary and sufficient conditions are given for the system $\mathscr{A}\cup\mathscr{B}$ to have an independent system of distinct representatives.

1. Introduction

Let $\mathscr{A}=\langle A_i : i\in I\rangle$ be a system of sets with index set I. We write

$$\mathscr{A}(J)=\bigcup\{A_i : i\in J\}$$

for any set $J\subset I$. A set $T\subset\mathscr{A}(I)$ is called a *transversal* of $\mathscr{A}$ if there is a bijection $\varphi : I\to T$ such that $\varphi(i)\in A_i$ $(\forall i\in I)$; we call φ a transversal function. We write $X \Subset Y$ to denote the fact that X is a finite subset of Y.

The classical result of transversal theory is the following theorem.

Theorem 1.1. *If $\mathscr{A}=\langle A_i : i\in I\rangle$ is a system of sets such that either* (a) *I is finite or* (b) *each A_i is finite, then $\mathscr{A}$ has a transversal if and only if*

$$|\mathscr{A}(J)|\geq|J| \quad (\forall J\Subset I). \tag{1}$$

This result was first proved by P. Hall [3] for condition (a) and later by M. Hall Jr. [2] for condition (b), and so the result is conveniently referred to as Hall's theorem.

Brualdi and Scrimger [1] (using properties of matroids) extended Theorem 1.1 by giving necessary and sufficient conditions for the existence of a transversal of a system having an arbitrary number of finite sets and a finite number of infinite sets. Woodall [5] proved an equivalent result (Theorem 1.2 below) using a simpler, more direct argument. In this note we generalize this result by giving necessary and sufficient conditions for such systems to have a transversal which is an independent set in some independence structure.

Let $\mathscr{A}=\langle A_i : i\in I\rangle$ be a system of finite sets and let $\mathscr{B}=\langle B_i : i\in I'\rangle$ be a finite system of arbitrary sets. We can assume without loss of generality that the index sets I, I' are disjoint and then we denote by $\mathscr{A}\cup\mathscr{B}$ the system $\langle C_i : i\in I\cup I'\rangle$, where $C_i=A_i$ $(i\in I)$ and $C_i=B_i$ $(i\in I')$. For any set X we define $\mathscr{A}^*(X)$ to be (i) the empty set if $\mathscr{A}$ has no transversal disjoint from X and (ii) the union $\bigcup\{\mathscr{A}(J) : J\Subset I, |\mathscr{A}(J)\setminus X|=|J|\}$ in the case when $\mathscr{A}$ does have a transversal disjoint from X. It is easily seen that, if $\mathscr{A}^*(X)$ is defined by (ii) (i.e. if $\mathscr{A}$ does have some transversal disjoint from X), then $\mathscr{A}^*(X)\setminus X$ is just the common intersection of all such transversals of $\mathscr{A}$. Woodall's formulation [5, Theorem 5] of the result referred to above is equivalent to the following.

Theorem 1.2. *The system $\mathscr{A}\cup\mathscr{B}$ has a transversal if and only if*

$$\mathscr{A} \text{ has a transversal} \tag{2}$$

and

$$\text{whenever } K\subset I' \text{ and } F\Subset\mathscr{B}(K), \text{ then} \tag{3}$$
$$|\mathscr{B}(K)\setminus(F\cup\mathscr{A}^*(F))|\geqslant|K|-|F|.$$

An independence structure is a pair $\mathscr{M}=\langle S,\mathscr{I}\rangle$, where S is a set and $\mathscr{I}$ is a non-empty set of subsets of S satisfying the following axioms:

(A_1) $Y\subset X\in\mathscr{I}\Rightarrow Y\in\mathscr{I}$ (hereditary);
(A_2) $X, Y\in\mathscr{I}$, $|X|<|Y|<\infty\Rightarrow(\exists y\in Y\setminus X)(X\cup\{y\}\in\mathscr{I})$ (exchange);
(A_3) $X\in\mathscr{I}\Leftrightarrow$ every finite subset of X is in $\mathscr{I}$ (finite character).

Members of $\mathscr{I}$ are called the *independent sets of $\mathscr{M}$*. It follows from Zorn's lemma that if $X\subset Y\subset S$ and $X\in\mathscr{I}$, then there is a maximal independent set $\bar{Y}$ such that $X\subset\bar{Y}\subset Y$. Also, in view of (A_2), every maximal independent subset (*basis*) of $Y\subset S$ has the same cardinality. Consequently, we can define a rank function $r(Y)=\max\{|Z| : Z\subset Y, Z\in\mathscr{I}\}$. In Section 2 we mention some other elementary facts about independence structures.

Rado [4] proved the following generalization of Theorem 1.1.

Theorem 1.3. *If $\mathscr{M}=\langle S,\mathscr{I}\rangle$ is an independence structure and $\mathscr{A}=\langle A_i : i\in I\rangle$ is a system of subsets of S having finite rank, then $\mathscr{A}$ has an independent transversal if and only if*

$$r(\mathscr{A}(J))\geqslant|J| \quad (\forall J\Subset I). \tag{4}$$

In particular, if $\mathscr{I}=\mathscr{P}(S)$ so that all subsets of S are "independent", then Theorem 1.3 reduces to Theorem 1.1.

If $\mathscr{M}=\langle S,\mathscr{I}\rangle$ is an independence structure and $X\subset S$ then $\mathscr{M}_X=\langle S\setminus X,\mathscr{I}_X\rangle$ is also an independence structure, where $Y\in\mathscr{I}_X\Leftrightarrow Y\subset S\setminus X$ and $Y\cup Z\in\mathscr{I}$ whenever Z is an independent subset of X. We denote the rank-function of $\mathscr{M}_X$

by r_X. For $X, Y \subset S$ we define

$$r(Y|X) = r_X(Y \setminus X).$$

Thus, if $X \cup Y$ has finite rank (in $\mathcal{M}$), then $r(Y \mid X) = r(X \cup Y) - r(X)$. We say that the two sets X, Y are *compatible* if $X \cap Y = \emptyset$ and if $X' \cup Y' \in \mathcal{I}$ whenever $X' \subset X$, $Y' \subset Y$ and X', $Y' \in \mathcal{I}$. Now suppose that $\mathcal{A} = \langle A_i : i \in I \rangle$ is a system of rank-finite subsets of S. It follows immediately from Theorem 1.3 (consider $\mathcal{M}_X$) that $\mathcal{A}$ has an independent transversal compatible with X if and only if

$$r(\mathcal{A}(J) \mid X) \geqslant |J| \quad (\forall J \Subset \mathcal{I}).$$

Now we define $\mathcal{A}^*_{\mathcal{M}}(X)$ to be (i) the empty set if $\mathcal{A}$ has no transversal compatible with X and (ii) the union $\bigcup \{\mathcal{A}(J) : J \Subset I, r(\mathcal{A}(J) \mid X) = |J|\}$ in the case when $\mathcal{A}$ does have an independent transversal compatible with X. Note that if $\mathcal{I} = \mathcal{P}(S)$, then $\mathcal{A}^*_{\mathcal{M}}(X)$ is just the set $\mathcal{A}^*(X)$ defined previously.

We are now able to state the generalization of Theorem 1.2 which is analogous to Rado's extension of Theorem 1.1.

Theorem 1.4. *Let $\mathcal{M} = \langle S, \mathcal{I} \rangle$ be an independence structure on S. Let $\mathcal{A} = \langle A_i : i \in I \rangle$ be a rank-finite system of subsets of S and $\mathcal{B} = \langle B_i : i \in I' \rangle$ a finite system of arbitrary subsets of S. Then $\mathcal{A} \cup \mathcal{B}$ has an independent transversal if and only if the following two conditions are satisfied:*

$$\mathcal{A} \text{ has an independent transversal} \tag{5}$$

and

whenever $K \subset I'$, $F \Subset \mathcal{B}(K)$ and $F \in \mathcal{I}$, then

$$r(\mathcal{B}(K) \mid F \cup \mathcal{A}^*_{\mathcal{M}}(F)) \geqslant |K| - |F|. \tag{6}$$

2. Some remarks on independence structures

A *circuit* of an independence structure $\mathcal{M} = \langle S, \mathcal{I} \rangle$ is a minimal dependent set, i.e. a set $C \subset S$ such that $C \notin \mathcal{I}$ but $C' \in \mathcal{I}$ for every proper subset C' of C. By (A_3), circuits are finite. The following two lemmas are well-known, but we include their short proofs.

Lemma 2.1. *If C_1, C_2 are distinct circuits and $x \in C_1 \cap C_2$, then there is a circuit $C \subset C_1 \cup C_2 \setminus \{x\}$.*

Proof. Suppose false. Then $C_1 \cup C_2 \setminus \{x\} \in \mathcal{I}$ and so every basis of $C_1 \cup C_2$ has cardinality $|C_1 \cup C_2| - 1$. Now $C_1 \cap C_2 \in \mathcal{I}$ and so there is a basis B of $C_1 \cup C_2$ which contains $C_1 \cap C_2$. But then B contains either C_1 or C_2, a contradiction.

Lemma 2.2. *If $X \subset S$ and $\bar{X}_1$, $\bar{X}_2$ are bases of X, then*

$$Z \cup \bar{X}_1 \in \mathcal{I} \Leftrightarrow Z \cup \bar{X}_2 \in \mathcal{I} \quad (\forall Z \subset S \setminus X).$$

Proof. Suppose on the contrary that $Z \cup \bar{X}_1 \in \mathcal{I}$, $Z \cup \bar{X}_2 \notin \mathcal{I}$. Then $Z \cup X$ contains a circuit C such that $Z \cap C \neq \emptyset$. We can assume that C is chosen so that $|X \cap C \setminus \bar{X}_1|$ is minimal. Now $C \not\subset Z \cup \bar{X}_1$ and so there is some $x \in X \cap C \setminus \bar{X}_1$. Also, since $\bar{X}_1$ is a basis, there is a circuit $C' \subset \{x\} \cup \bar{X}_1$. By Lemma 2.1 it follows that there is a circuit $C'' \subset C \cup C' \setminus \{x\}$ and this contradicts the minimality condition assumed for C.

It follows from Lemma 2.2 that $\mathcal{I}_X = \{Y \subset S \setminus X : Y \cup \bar{X} \in \mathcal{I}\}$, where $\bar{X}$ is any basis of X. Also, for disjoint sets $X, Y \subset S$, $(\mathcal{I}_X)_Y = \mathcal{I}_{X \cup Y}$.

We also need the following lemma.

Lemma 2.3. *Let $\mathcal{A} = \langle A_i : i \in I \rangle$ be a system of subsets of S which has an independent transversal. If $J_1, J_2 \Subset I$ and*

$$r(\mathcal{A}(J_i)) = |J_i| \quad (i = 1, 2),$$

then

$$r(\mathcal{A}(J_1 \cup J_2)) = |J_1 \cup J_2|.$$

Proof. Let φ be an independent transversal function of $\mathcal{A}$. Then $\varphi(J_i)$ is a basis of $\mathcal{A}(J_i)$ $(i = 1, 2)$. Also $\varphi(J_1 \cup J_2)$ is an independent subset of cardinality $|J_1 \cup J_2|$. Suppose $r(\mathcal{A}(J_1 \cup J_2)) > |J_1 \cup J_2|$. Then there is $x \in \mathcal{A}(J_1 \cup J_2) \setminus \varphi(J_1 \cup J_2)$ such that $\{x\} \cup \varphi(J_1 \cup J_2) \in \mathcal{I}$. There is $i \in \{1, 2\}$ such that $x \in \mathcal{A}(J_i)$ and now we have the contradiction that $\{x\} \cup \varphi(J_i)$ is an independent subset of $\mathcal{A}(J_i)$ of cardinality $|J_i| + 1$.

We remarked earlier that if the system of finite sets $\mathcal{A}$ has a transversal disjoint from X, then $\mathcal{A}^*(X) = \bigcap\{T : T$ is a transversal of $\mathcal{A}$ disjoint from $X\}$. We conclude this section by noting an analogous description of $\mathcal{A}^*_{\mathcal{M}}(X)$. A set Y is said to *depend upon* $X \subset S$ if $\{x\} \cup \bar{X} \notin \mathcal{I}$ whenever $x \in Y \setminus X$ and $\bar{X}$ is a basis of X. We write $Y \mid X$ if Y depends upon X and $Y \nmid X$ if this is false.

Lemma 2.4. *Let $\mathcal{M} = \langle S, \mathcal{I} \rangle$ be an independence structure on S, $X \subset S$ and let $\mathcal{A} = \langle A_i : i \in I \rangle$ be a system of rank-finite subsets of S which has an independent transversal compatible with X. Then $\mathcal{A}^*_{\mathcal{M}}(X) = \bigcup\{A_i : i \in I$ and $A_i \mid T \cup X$ for every independent transversal T of $\mathcal{A}$ compatible with $X\}$.*

Proof. Suppose $x \in \mathcal{A}^*_{\mathcal{M}}(X)$. Then there are $J \Subset I$ and $i \in J$ such that $r(\mathcal{A}(J) \mid X) = |J|$ and $x \in A_i$. Let T be any independent transversal of $\mathcal{A}$ compatible with X. Then $T \cap \mathcal{A}(J)$ is a basis for $\mathcal{A}(J) \setminus X$ in $\mathcal{M}_X$ and hence $A_i \mid T \cup X$.

Now suppose that $i \in I$ and $A_i \mid T \cup X$ for every independent transversal T of $\mathcal{A}$ compatible with X. By hypothesis $A_i \nmid X$. Let $\bar{A}_i$ be a basis for $A_i \setminus X$ in $\mathcal{M}_X$. Then for each $x \in \bar{A}_i$, the system $\mathcal{A}$ has no transversal compatible with $X \cup \{x\}$ (otherwise there is an independent transversal T such that $\{x\} \nmid T \cup X$). Therefore, by Theorem 1.3, there is $J_x \Subset I$ such that $r(\mathcal{A}(J_x) \mid X \cup \{x\}) < |J_x|$. Since

$r(\mathscr{A}(J_x) \mid X) \geq |J_x|$, it follows that

$$r(\mathscr{A}(J_x) \mid X) = r(\mathscr{A}(J_x) \cup \{x\} \mid X) = |J_x|.$$

Put $J^* = \bigcup \{J_x : x \in \bar{A}_i\}$. By Lemma 2.3, $r(\mathscr{A}(J^*) \mid X) = |J^*|$ and $r(\mathscr{A}(J^*) \cup A_i \mid X) = |J^*|$. Since $\mathscr{A}$ has an independent transversal compatible with X, it follows that $i \in J^*$ and hence that $A_i \subset \mathscr{A}^*_{\mathscr{M}}(X)$.

3. Proof of Theorem 1.4

3.1. *Necessity*

Suppose $\mathscr{A} \cup \mathscr{B}$ has an independent transversal T. Then (5) holds trivially. It remains to verify that (6) also holds.

Let $K \subset I'$, $F \Subset S$, $F \in \mathscr{I}$. Let $\varphi : I \cup I' \to T$ be a transversal function. In order to prove (6) it will be enough to show that $r(\varphi(K) \mid F \cup \mathscr{A}^*_{\mathscr{M}}(F)) \geq |K| - |F|$ or, equivalently, that

$$r_F(\varphi(K) \setminus F \mid \mathscr{A}^*_{\mathscr{M}}(F) \setminus F) \geq |K| - |F|. \tag{7}$$

If $\mathscr{A}^*_{\mathscr{M}}(F) = \emptyset$ then (7) holds trivially and so we may assume that $\mathscr{A}^*_{\mathscr{M}}(F) \neq \emptyset$.

Let $r_0 = r_F(\varphi(K) \setminus F \mid \mathscr{A}^*_{\mathscr{M}}(F) \setminus F)$. Then for each $(r_0 + 1)$-element subset X of $\varphi(K) \setminus (F \cup \mathscr{A}^*_{\mathscr{M}}(F))$ there is a finite set $G(X) \subset \mathscr{A}^*_{\mathscr{M}}(F) \setminus F$ such that $G(X) \in \mathscr{I}_F$ and $X \cup G(X) \notin \mathscr{I}_F$. Put $G^* = \bigcup \{G(X) : X \subset \varphi(K) \setminus (F \cup \mathscr{A}^*_{\mathscr{M}}(F)),\ |X| = r_0 + 1\}$. Then $r_F(\varphi(K) \setminus (F \cup \mathscr{A}^*_{\mathscr{M}}(F)) \mid G^*)$ is also equal to r_0. Now, by Lemma 2.3 and the definition of $\mathscr{A}^*_{\mathscr{M}}(F)$, there is $J^* \Subset I$ such that $r_F(\mathscr{A}(J^*) \setminus F) = |J^*|$ and such that $G^* \cup (\varphi(K) \cap \mathscr{A}^*_{\mathscr{M}}(F)) \subset \mathscr{A}(J^*)$. Since $T' = \varphi(K \cup J^*) \in \mathscr{I}$, it follows by the exchange property (A_2) that there is $T'_1 \subset T' \setminus F$ such that $T'_1 \cup F \in \mathscr{I}$ and $|T'_1| \geq |T'| - |F|$. Also, since $T'_1 \in \mathscr{I}_F$ and $r_F(\mathscr{A}(J^*) \setminus F) = |J^*|$, it follows, again by ($A_2$), that there is $T'_2 \subset T'_1 \setminus \mathscr{A}(J^*)$ such that $T'_2 \in \mathscr{I}_{F \cup \mathscr{A}(J^*)}$ and $|T'_2| \geq |T'_1| - |J^*|$. Now $T'_2 \subset \varphi(K) \setminus (F \cup \mathscr{A}^*_{\mathscr{M}}(F))$ and so

$$\begin{aligned} r_0 &= r_F(\varphi(K) \setminus (F \cup \mathscr{A}^*_{\mathscr{M}}(F)) \mid G^*) \\ &\geq r_F(\varphi(K) \setminus (F \cup \mathscr{A}^*_{\mathscr{M}}(F)) \mid \mathscr{A}(J^*) \setminus F) \\ &\geq |T'_2| \geq |T'| - |F| - |J^*| = |K| - |F|. \end{aligned}$$

This proves (7).

3.2. *Sufficiency*

We now assume that (5) and (6) hold and deduce that $\mathscr{A} \cup \mathscr{B}$ has an independent transversal.

Let $K \subset I'$. We will define a set $F(K) \Subset \mathscr{B}(K)$ such that $F(K) \in \mathscr{I}$ and such that $\mathscr{A}$ has an independent transversal compatible with $F(K)$. Put $F_0 = \emptyset$. Let $0 \leq l < k = |K|$ and suppose we have already chosen a set $F_l \subset \mathscr{B}(K)$ such that $F_l \in \mathscr{I}$, $|F_l| = l$ and such that $\mathscr{A}$ has an independent transversal T_l compatible with F_l. Note that this is true for $l=0$ in view of (5). By (6), $r(\mathscr{B}(K) \mid F_l \cup \mathscr{A}^*_{\mathscr{U}}(F_l)) > 0$. Hence there is an element $x_l \in \mathscr{B}(K) \setminus (F_l \cup \mathscr{A}^*_{\mathscr{U}}(F_l))$ such that x_l does not depend upon $F_l \cup \mathscr{A}^*_{\mathscr{U}}(F_l)$. Put $F_{l+1} = F_l \cup \{x_l\}$. Suppose $\mathscr{A}$ has no independent transversal compatible with F_{l+1}. Then there is some $J \Subset I$ such that $r(\mathscr{A}(J) \mid F_{l+1}) < |J|$. Then $r(\mathscr{A}(J) \mid F_l) \leqslant |J|$ and since $B_l = I_l \cap \mathscr{A}(J) \in \mathscr{I}_{l}$ and has cardinality $\geqslant |J|$, it follows that $|B_l| = |J|$, and $\mathscr{A}(J) \subset \mathscr{A}^*_{\mathscr{U}}(F_l)$. Now $B_l \cup F_{l+1} \notin \mathscr{I}$ and so $x_l \mid B_l \cup F_l$ and hence $x_l \mid F_l \cup \mathscr{A}^*_{\mathscr{U}}(F_l)$, a contradiction. Thus $F_{l+1} \in \mathscr{I}$, $|F_{l+1}| = l+1$ and $\mathscr{A}$ does have an independent transversal compatible with F_{l+1}. By induction this defines F_l for $0 \leqslant l \leqslant k$ and then $F(K) = F_k$ has the desired properties.

Now put $F^* = \bigcup\{F(K) : K \subset I'\}$ and let $\mathscr{B}' = (B_i \cap F^* : i \in I')$. We want to show that the system $\mathscr{A} \cup \mathscr{B}'$ of rank-finite subsets of S satisfies Rado's condition (4). This will imply that $\mathscr{A} \cup \mathscr{B}'$ has an independent transversal and hence so also has $\mathscr{A} \cup \mathscr{B}$. Let $K \Subset I'$, $J \Subset I$. Then

$$r(\mathscr{B}'(K) \cup \mathscr{A}(J)) \geqslant r(F(K) \cup \varphi_K(J)) = |K| + |J|,$$

where $\varphi_K : I \to \mathscr{A}(I)$ is a transversal function such that $\varphi_K(I)$ is an independent transversal of $\mathscr{A}$ compatible with $F(K)$.

4. A remark about condition (6)

In proving the necessity part of Theorem 1.4 we actually proved that the stronger condition

$$(\forall K \Subset I')(\forall X \subset S) \quad (r(\mathscr{B}(K) \mid X \cap_{\mathscr{A}} \mathscr{A}^*_{\mathscr{U}}(X)) \geqslant |K| - |X| \tag{6'}$$

holds. Also in proving the sufficiency part, we only really need the weaker condition

$$(\forall K \Subset I')(\forall F \subset \mathscr{B}(K)) \quad (|F| = |K| - 1 \Rightarrow r(\mathscr{B}(K) \mid F \cup \mathscr{A}^*_{\mathscr{U}}(F)) \geqslant |K| - |F|). \tag{6''}$$

Thus (5), together with any of (6), (6′), (6″) are equivalent pairs of conditions.

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